

## LIE DERIVATIONS OF DUAL EXTENSIONS

YANBO LI AND FENG WEI

ABSTRACT. Let  $K$  be a field and  $\Gamma$  a finite quiver without oriented cycles. Let  $\Lambda$  be the path algebra  $K(\Gamma, \rho)$  and let  $\mathcal{D}(\Lambda)$  be the dual extension of  $\Lambda$ . In this paper, we prove that each Lie derivation of  $\mathcal{D}(\Lambda)$  is of the standard form.

## 1. INTRODUCTION

In the study of the representation theory of quasi-hereditary algebras, Xi [18] defined dual extensions of algebras without oriented cycles. Roughly speaking, these algebras  $A$  are constructed by adding to the ordinary quiver (without oriented cycles) of a given algebra  $B$  a reverse arrow for any original arrow and extending the relations in a suitable way to this extended quiver. They are a class of finite dimensional quasi-hereditary algebras and they were detailedly investigated in [6, 8, 19] by Deng and Xi. A dual extension algebra is a BGG-algebra in the sense of R. Irving [10], that is, a quasi-hereditary algebra with a duality which fixes all simple modules. A much common more general construction, the twisted doubles, were studied in [7, 17, 20] by Deng, Koenig and Xi.

Derivations and Lie derivations of associative algebras, as classical linear mappings, play significant roles in various mathematical areas, such as in Lie theory, matrix theory, noncommutative algebras and operator algebras. Let  $\mathcal{R}$  be a commutative ring with identity,  $\mathcal{A}$  be a unital algebra over  $\mathcal{R}$  and  $\mathcal{Z}(\mathcal{A})$  be the center of  $\mathcal{A}$ . We write  $[a, b] = ab - ba$  for all  $a, b \in \mathcal{A}$ . Let  $\Theta: \mathcal{A} \rightarrow \mathcal{A}$  be a linear mapping. We call  $\Theta$  an (*associative*) *derivation* if

$$\Theta(ab) = \Theta(a)b + a\Theta(b)$$

for all  $a, b \in \mathcal{A}$ . Further,  $\Theta$  is called a *Lie derivation* if

$$\Theta([a, b]) = [\Theta(a), b] + [a, \Theta(b)]$$

for all  $a, b \in \mathcal{A}$ . It is clear that every associative derivation is a Lie derivation. But, the converse statement is not true in general. Moreover, if  $D: \mathcal{A} \rightarrow \mathcal{A}$  is an associative derivation and  $\Delta: \mathcal{A} \rightarrow \mathcal{Z}(\mathcal{A})$  is a linear mapping such that  $\Delta([a, b]) = 0$  for all  $a, b \in \mathcal{A}$ , then the mapping

$$\Theta = D + \Delta, \quad (\spadesuit)$$

is a Lie derivation. We shall say that a Lie derivation is *standard* in the case where it can be expressed in the preceding form. We call  $\Theta$  a *Jordan derivation* if

$$\Theta(a \circ b) = \Theta(a) \circ b + a \circ \Theta(b)$$

---

2000 *Mathematics Subject Classification.* 15A78, 16W25, 47C05.

*Key words and phrases.* Dual extension, generalized matrix algebra, Lie derivation,

The first author of this work is supported by Fundamental Research Funds for the Central Universities (N110423007).

for all  $a, b \in \mathcal{A}$ . Of course every associative derivation is a Jordan derivation, while the converse statement is not always true. We shall say that a Jordan derivation is *standard* if it is an associative derivation.

A common and popular problem in the study of Lie derivations is whether they have the above mentioned standard form. Equivalently, how every Lie derivation is approximate to a derivation to the utmost extent. The first result in this aspect is due to Martindale, who proved that each Lie derivation of a prime ring satisfying some conditions is of the standard form in [15]. Alaminos et al [1] showed that every Lie derivation on the full matrix algebra over a field of characteristic zero has the standard form. Cheung [4] considered Lie derivations of triangular algebras and gave a sufficient and necessary condition which enables every Lie derivation to be standard. Benkovic studied the structure of Lie derivations from a triangular algebra into its bi-module in [3]. The description of standard form on Lie triple derivations of triangular algebras were obtained by Xiao and Wei in [22]. Recently, the current authors and Xiao investigated the associative-type, Lie-type and Jordan-type linear mappings of generalized matrix algebras. For details, we refer the reader to [11, 12, 14, 21].

Path algebras of quivers naturally appear in the study of tensor algebras of bi-modules over semisimple algebras. It is well known that any finite dimensional basic  $K$ -algebra is given by a quiver with relations when  $K$  is an algebraically closed field. In [9], Guo and Li studied the Lie algebra of differential operators on a path algebra  $K\Gamma$  and related this Lie algebra to the algebraic and combinatorial properties of the path algebra  $K\Gamma$ . In [12], the current authors studied Lie derivations of a class of path algebras of quivers without oriented cycles, which can be viewed as one-point extensions. It was proved that in this case each Lie derivation is of the standard form. Moreover, the standard decomposition is unique. On the other hand, we remark that its dual extension algebra of arbitrary finite dimensional algebra inherit many wonderful properties from the given algebra. Then for the path algebra of a finite quiver without oriented cycles, it is natural to ask whether all Lie derivations on the dual extension algebra are of the standard form. We will give a positive answer in this paper. More precisely, the main result of this paper is

**Theorem.** *Let  $K$  be a field of characteristic not 2. Let  $(\Gamma, \rho)$  be a finite quiver without oriented cycles. Then each Lie derivation on the dual extension of the path algebra  $K(\Gamma, \rho)$  is of the standard form ( $\spadesuit$ ). Moreover, the standard decomposition is unique.*

Jordan derivations, another important class of linear mappings on dual extension algebras, has been characterized in [13], where we show that every Jordan derivation on dual extension algebras is also of the standard form.

Note that each associative algebra with non trivial idempotents is isomorphic a generalized matrix algebra. Recently, Du and Wang [5] studied Lie derivations of generalized matrix algebras with bi-modules  $M$  being faithful. Although the methods of matrix algebras is also employed in our current work, we prefer to the assumptions without faithful conditions. When Cheung [4] investigated Lie derivations of triangular algebras, the faithful assumption is not needed. In this sense,

Section 3 of this paper is a natural generalization of Cheung's work. Simultaneously, our work is an attempt to deal with path algebras of quivers with oriented cycles. So this article is also a continuation and development of [12].

The paper is organized as follows. After a rapid review of some needed preliminaries in Section 2, we characterize Lie derivations of generalized matrix algebras in Section 3. We study Lie derivations of dual extensions in Section 4, where the main result of this paper will be eventually obtained.

## 2. DUAL EXTENSION

Let us first recall the definition of dual extensions of path algebras which were introduced by Xi [18]. Moreover, in order to use the methods of matrix algebras, we will also give some descriptions of dual extensions from the point of view of generalized matrix algebras. This kind of algebra was introduced by Morita in [16], where the author studied Morita duality theory of modules and its applications to Artinian algebras. Let us begin with the definition of generalized matrix algebras.

**2.1. Generalized matrix algebras.** The definition of generalized matrix algebras is given by a Morita context. Let  $\mathcal{R}$  be a commutative ring with identity. A Morita context consists of two  $\mathcal{R}$ -algebras  $A$  and  $B$ , two bimodules  ${}_A M_B$  and  ${}_B N_A$ , and two bimodule homomorphisms called the pairings  $\Phi_{MN} : M \otimes_B N \rightarrow A$  and  $\Psi_{NM} : N \otimes_A M \rightarrow B$  satisfying the following commutative diagrams:

$$\begin{array}{ccc} M \otimes_B N \otimes_A M & \xrightarrow{\Phi_{MN} \otimes I_M} & A \otimes_A M \\ \downarrow I_M \otimes \Psi_{NM} & & \downarrow \cong \\ M \otimes_B B & \xrightarrow{\cong} & M \end{array}$$

and

$$\begin{array}{ccc} N \otimes_A M \otimes_B N & \xrightarrow{\Psi_{NM} \otimes I_N} & B \otimes_B N \\ \downarrow I_N \otimes \Phi_{MN} & & \downarrow \cong \\ N \otimes_A A & \xrightarrow{\cong} & N. \end{array}$$

Let us write this Morita context as  $(A, B, {}_A M_B, {}_B N_A, \Phi_{MN}, \Psi_{NM})$ . If  $(A, B, {}_A M_B, {}_B N_A, \Phi_{MN}, \Psi_{NM})$  is a Morita context, then the set

$$\begin{bmatrix} A & M \\ N & B \end{bmatrix} = \left\{ \begin{bmatrix} a & m \\ n & b \end{bmatrix} \middle| a \in A, m \in M, n \in N, b \in B \right\}$$

form an  $\mathcal{R}$ -algebra under matrix-like addition and matrix-like multiplication. There is no constraint condition concerning the bimodules  $M$  and  $N$ . Of course, they

probably equal to zeros. Such an  $\mathcal{R}$ -algebra is called a *generalized matrix algebra* of order 2 and is usually denoted by  $\mathcal{G} = \begin{bmatrix} A & M \\ N & B \end{bmatrix}$ . Its center  $\mathcal{Z}(\mathcal{G})$  is

$$\mathcal{Z}(\mathcal{G}) = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \middle| a \in \mathcal{Z}(A), b \in \mathcal{Z}(B), am = mb, na = bn, \forall m \in M, n \in N \right\}.$$

Thus we have two natural  $\mathcal{R}$ -linear projections  $\pi_A : \mathcal{G} \rightarrow A$  and  $\pi_B : \mathcal{G} \rightarrow B$  by

$$\pi_A : \begin{bmatrix} a & m \\ n & b \end{bmatrix} \mapsto a \quad \text{and} \quad \pi_B : \begin{bmatrix} a & m \\ n & b \end{bmatrix} \mapsto b.$$

Then  $\pi_A(\mathcal{Z}(\mathcal{G}))$  is a subalgebra of  $\mathcal{Z}(A)$  and that  $\pi_B(\mathcal{Z}(\mathcal{G}))$  is a subalgebra of  $\mathcal{Z}(B)$ . If  $M$  is faithful as a right  $B$ -module and as left  $A$ -module, then for every element  $a \in \pi_A(\mathcal{Z}(\mathcal{G}))$ , there exists a unique  $b \in \pi_B(\mathcal{Z}(\mathcal{G}))$ , which is denoted by  $\varphi(a)$ , such that  $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \in \mathcal{Z}(\mathcal{G})$ . It is easy to verify that the mapping  $\varphi : \pi_A(\mathcal{Z}(\mathcal{G})) \rightarrow \pi_B(\mathcal{Z}(\mathcal{G}))$  is an algebraic isomorphism such that  $am = m\varphi(a)$  and  $na = \varphi(a)n$  for all  $a \in \pi_A(\mathcal{Z}(\mathcal{G})), m \in M, n \in N$ .

**Remark 2.1.** Any unital  $\mathcal{R}$ -algebra  $\mathcal{A}$  with nontrivial idempotents is isomorphic to a generalized matrix algebra. In fact, suppose that there exists a nontrivial idempotent  $e \in \mathcal{A}$ . We construct the following *natural generalized matrix algebra*:

$$\begin{aligned} \mathcal{G} &= \begin{bmatrix} e\mathcal{A}e & e\mathcal{A}(1-e) \\ (1-e)\mathcal{A}e & (1-e)\mathcal{A}(1-e) \end{bmatrix} \\ &= \left\{ \begin{bmatrix} eae & ec(1-e) \\ (1-e)de & (1-e)b(1-e) \end{bmatrix} \middle| a, b, c, d \in \mathcal{A} \right\}. \end{aligned}$$

It is easy to check that the  $\mathcal{R}$ -linear mapping

$$\begin{aligned} \xi : \mathcal{A} &\longrightarrow \mathcal{G} \\ a &\longmapsto \begin{bmatrix} eae & ea(1-e) \\ (1-e)ae & (1-e)a(1-e) \end{bmatrix} \end{aligned}$$

is an isomorphism from  $\mathcal{A}$  to  $\mathcal{G}$ .

**2.2. Dual extension of path algebras.** Recall that a *finite quiver*  $\Gamma$  is an oriented graph with the set of vertices  $\Gamma_0$  and the set of arrows between vertices  $\Gamma_1$  being both finite. For an arrow  $\alpha$ , we write  $s(\alpha) = i$  and  $e(\alpha) = j$  if it is from the vertex  $i$  to the vertex  $j$ . A *sink* is a vertex without arrows beginning at it and a *source* is a vertex without arrows ending at it. A *nontrivial path* in  $\Gamma$  is an ordered sequence of arrows  $p = \alpha_n \cdots \alpha_1$  such that  $e(\alpha_m) = s(\alpha_{m+1})$  for each  $1 \leq m < n$ . Define  $s(p) = s(\alpha_1)$  and  $e(p) = e(\alpha_n)$ . A *trivial path* is the symbol  $e_i$  for each  $i \in \Gamma_0$ . In this case, we set  $s(e_i) = e(e_i) = i$ . A nontrivial path  $p$  is called an *oriented cycle* if  $s(p) = e(p)$ . Denote the set of all paths by  $\mathcal{P}$ .

Let  $K$  be a field and  $\Gamma$  be a quiver. Then the path algebra  $K\Gamma$  is the  $K$ -algebra generated by the paths in  $\Gamma$  and the product of two paths  $x = \alpha_n \cdots \alpha_1$  and  $y = \beta_t \cdots \beta_1$  is defined by

$$xy = \begin{cases} \alpha_n \cdots \alpha_1 \beta_t \cdots \beta_1, & e(y) = s(x); \\ 0, & \text{otherwise.} \end{cases}$$

Clearly,  $K\Gamma$  is an associative algebra with the identity  $1 = \sum_{i \in \Gamma_0} e_i$ , where  $e_i (i \in \Gamma_0)$  are pairwise orthogonal primitive idempotents of  $K\Gamma$ .

A *relation*  $\sigma$  on a quiver  $\Gamma$  over a field  $K$  is a  $K$ -linear combination of paths  $\sigma = \sum_{i=1}^n k_i p_i$ , where  $k_i \in K$  and

$$e(p_1) = \cdots = e(p_n), \quad s(p_1) = \cdots = s(p_n).$$

Moreover, the number of arrows in each path is assumed to be at least 2. Let  $\rho$  be a set of relations on  $\Gamma$  over  $K$ . The pair  $(\Gamma, \rho)$  is called a *quiver* with relations over  $K$ . Denote by  $\langle \rho \rangle$  the ideal of  $K\Gamma$  generated by the set of relations  $\rho$ . The  $K$ -algebra  $K(\Gamma, \rho) = K\Gamma / \langle \rho \rangle$  is always associated with  $(\Gamma, \rho)$ . For arbitrary element  $x \in K\Gamma$ , write by  $\bar{x}$  the corresponding element in  $K(\Gamma, \rho)$ . We often write  $\bar{x}$  as  $x$  if this is not misled or confused. We refer the reader to [2] for the basic facts of path algebras.

Let  $\Lambda = K(\Gamma, \rho)$ , where  $\Gamma$  is a finite quiver. Let  $\Gamma^*$  to be a quiver whose vertex set is  $\Gamma_0$  and

$$\Gamma_1^* = \{\alpha^* : i \rightarrow j \mid \alpha : j \rightarrow i \text{ is an arrow in } \Gamma_1\}.$$

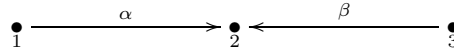
Let  $p = \alpha_n \cdots \alpha_1$  is a path in  $\Gamma$ . Write the path  $\alpha_1^* \cdots \alpha_n^*$  in  $\Gamma^*$  by  $p^*$ . Define  $\mathcal{D}(\Lambda)$  to be the path algebra of the quiver  $(\Gamma_0, \Gamma_1 \cup \Gamma_1^*)$  with relations

$$\rho \cup \rho^* \cup \{\alpha\beta^* \mid \alpha, \beta \in \Gamma_1\}.$$

If  $\Gamma$  has no oriented cycles, then  $\mathcal{D}(\Lambda)$  is called the *dual extension* of  $\Lambda$ . It is a BGG-algebra in the sense of [10]. Clearly, if  $|\Gamma_0| = 1$ , then the algebra is trivial. Let us assume that  $|\Gamma_0| \geq 2$  from now on. It is helpful to point out that in this case,  $\mathcal{D}(\Lambda)$  has non-trivial idempotents. In view of Remark 2.1,  $\mathcal{D}(\Lambda)$  is isomorphic to a generalized matrix algebra  $\mathcal{G} = \begin{bmatrix} A & M \\ N & B \end{bmatrix}$ .

Let us take the nontrivial idempotent to be  $e_i$ , where  $i$  is a source of  $\Gamma$ . According to the definition of dual extension, it is easy to verify that the pairings  $\Phi_{MN} = 0$  and  $\Psi_{NM} \neq 0$ . If  $M \neq 0$ , then  $N \neq 0$ . Moreover, it is helpful to point out that  $M$  need not to be faithful as left  $A$ -module or as right  $B$ -module. Let us illustrate two examples in below.

**Example 2.2.** Let  $\Gamma$  be a quiver as follows

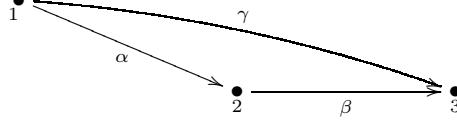


and let  $\Lambda = K\Gamma$ . The dual extension  $\mathcal{D}(\Lambda)$  has a basis

$$\{e_1, e_2, e_3, \alpha, \beta, \alpha^*, \beta^*, \alpha^*\alpha, \beta^*\alpha, \beta^*\beta, \alpha^*\beta\}.$$

Taking the nontrivial idempotent to be  $e_1$ , then  $\mathcal{D}(\Lambda)$  is isomorphic to the generalized matrix algebra  $\mathcal{G} = \begin{bmatrix} A & M \\ N & B \end{bmatrix}$ , where  $A$  has a basis  $\{e_2, e_3, \beta, \beta^*, \beta^*\beta\}$ ,  $B$  has a basis  $\{e_1, \alpha^*\alpha\}$ ,  $M$  has a basis  $\{\alpha, \beta^*\alpha\}$  and  $N$  has a basis  $\{\alpha^*, \alpha^*\beta\}$ . It follows from  $\beta\alpha = 0$  and  $\beta\beta^*\alpha = 0$  that  $\beta \in \text{anni}({}_A M)$ . That is,  $M$  is not faithful as left  $A$ -module. It is easy to check that  $\alpha^*\alpha \in \text{anni}(M_B)$ . This implies that  $M$  is not faithful as right  $B$ -module. Similarly, we obtain  $\alpha^*\alpha \in \text{anni}({}_B N)$  and  $\beta^*\beta \in \text{anni}(N_A)$ . That is,  $N$  is neither a faithful left  $B$ -module nor a faithful right  $A$ -module.

**Example 2.3.** Let  $\Gamma$  be a quiver as follows



and let  $\Lambda = K\Gamma$ . Taking the nontrivial idempotent to be  $e_1$ , then the dual extension  $\mathcal{D}(\Lambda) \simeq \mathcal{G}(A, M, N, B)$ , where  $A$  has a basis  $\{e_2, e_3, \beta, \beta^*, \beta^*\beta\}$ ,  $M$  has a basis  $\{\alpha, \beta^*\beta\alpha, \beta^*\gamma, \beta\alpha, \gamma\}$ ,  $N$  has a basis  $\{\alpha^*, \gamma^*\beta, \alpha^*\beta^*\beta, \alpha^*\beta^*, \gamma^*\}$ ,  $B$  has a basis  $\{e_1, \alpha^*\alpha, \gamma^*\gamma, \alpha^*\beta^*\gamma, \gamma^*\beta\alpha, \alpha^*\beta^*\beta\alpha\}$ . Clearly,  $e_2\alpha \neq 0$ ,  $e_3\beta\alpha \neq 0$ . Then  $e_2, e_3 \notin \text{anni}(M)$ . Similarly,  $\beta\alpha \neq 0$  implies that  $\beta \notin \text{anni}(M)$ ;  $\beta^*\beta\alpha \neq 0$  implies that  $\beta^* \notin \text{anni}(M)$  and  $\beta^*\beta \notin \text{anni}(M)$ . Hence  $M$  is faithful as a left  $A$ -module. On the other hand, it is easy to check that  $\alpha^*\beta^*\beta\alpha \in \text{anni}(M_B)$ . Thus  $M$  is not faithful as a right  $B$ -module. Similarly, we know that  $N$  is faithful as a right  $A$ -module, while it is not faithful as a left  $B$ -module.

**Remark 2.4.** In order to study global dimension of dual extensions, one more general definition of dual extension algebras was posed by Xi [19]. We omit the details here because it will not be used in our current work.

### 3. LIE DERIVATIONS OF GENERALIZED MATRIX ALGEBRAS

In Section 2 we have pointed out that the dual extension of a path algebra can be viewed as a generalized matrix algebra. In order to study Lie derivations of dual extension algebras, it is necessary to provide some basic facts concerning Lie derivations of generalized matrix algebras. In this section, we will give a sufficient and necessary condition which enable every Lie derivation to be standard ( $\spadesuit$ ).

From now on, we always assume that all algebras and bimodules are 2-torsion free. Note that the forms of derivations and Lie derivations of a generalized matrix algebra have already been described in [11].

**Lemma 3.1.** [11, Proposition 4.1] *Let  $\Theta_{\text{Lied}}$  be a Lie derivation of a generalized matrix algebra  $\mathcal{G} = \begin{bmatrix} A & M \\ N & B \end{bmatrix}$ . Then  $\Theta_{\text{Lied}}$  is of the form*

$$\begin{aligned} & \Theta_{\text{Lied}} \left( \begin{bmatrix} a & m \\ n & b \end{bmatrix} \right) \\ &= \begin{bmatrix} \delta_1(a) - mn_0 - m_0n + \delta_4(b) & am_0 - m_0b + \tau_2(m) \\ n_0a - bn_0 + \nu_3(n) & \mu_1(a) + n_0m + nm_0 + \mu_4(b) \end{bmatrix}, \\ & \forall \begin{bmatrix} a & m \\ n & b \end{bmatrix} \in \mathcal{G}, \end{aligned}$$

where  $m_0 \in M, n_0 \in N$  and

$$\begin{aligned} \delta_1 : A &\longrightarrow A, & \delta_4 : B &\longrightarrow \mathcal{Z}(A) & \tau_2 : M &\longrightarrow M, \\ \nu_3 : N &\longrightarrow N & \mu_1 : A &\longrightarrow \mathcal{Z}(B), & \mu_4 : B &\longrightarrow B \end{aligned}$$

are all  $\mathcal{R}$ -linear mappings satisfying the following conditions:

- (1)  $\delta_1$  is a Lie derivation of  $A$  and  $\delta_1(mn) = \delta_4(nm) + \tau_2(m)n + m\nu_3(n)$ ;
- (2)  $\mu_4$  is a Lie derivation of  $B$  and  $\mu_4(nm) = \mu_1(mn) + n\tau_2(m) + \nu_3(n)m$ ;
- (3)  $\delta_4([b, b']) = 0$  for all  $b, b' \in B$  and  $\mu_1([a, a']) = 0$  for all  $a, a' \in A$ ;
- (4)  $\tau_2(am) = a\tau_2(m) + \delta_1(a)m - m\mu_1(a)$  and  $\tau_2(mb) = \tau_2(m)b + m\mu_4(b) - \delta_4(b)m$ ;

$$(5) \quad \nu_3(na) = \nu_3(n)a + n\delta_1(a) - \mu_1(a)n \text{ and } \nu_3(bn) = b\nu_3(n) + \mu_4(b)n - n\delta_4(b).$$

**Lemma 3.2.** [11, Proposition 4.2] *An additive mapping  $\Theta_d$  is a derivation of  $\mathcal{G}$  if and only if  $\Theta_d$  has the form*

$$\begin{aligned} & \Theta_d \left( \begin{bmatrix} a & m \\ n & b \end{bmatrix} \right) \\ &= \begin{bmatrix} \delta_1(a) - mn_0 - m_0n & am_0 - m_0b + \tau_2(m) \\ n_0a - bn_0 + \nu_3(n) & n_0m + nm_0 + \mu_4(b) \end{bmatrix}, \\ & \forall \begin{bmatrix} a & m \\ n & b \end{bmatrix} \in \mathcal{G}, \end{aligned}$$

where  $m_0 \in M, n_0 \in N$  and

$$\begin{aligned} \delta_1 : A &\longrightarrow A, & \tau_2 : M &\longrightarrow M, & \tau_3 : N &\longrightarrow M, \\ \nu_2 : M &\longrightarrow N, & \nu_3 : N &\longrightarrow N, & \mu_4 : B &\longrightarrow B \end{aligned}$$

are all  $\mathcal{R}$ -linear mappings satisfying the following conditions:

- (1)  $\delta_1$  is a derivation of  $A$  with  $\delta_1(mn) = \tau_2(m)n + m\nu_3(n)$ ;
- (2)  $\mu_4$  is a derivation of  $B$  with  $\mu_4(nm) = n\tau_2(m) + \nu_3(n)m$ ;
- (3)  $\tau_2(am) = a\tau_2(m) + \delta_1(a)m$  and  $\tau_2(mb) = \tau_2(m)b + m\mu_4(b)$ ;
- (4)  $\nu_3(na) = \nu_3(n)a + n\delta_1(a)$  and  $\nu_3(bn) = b\nu_3(n) + \mu_4(b)n$ ;

In [4], Cheung gave a necessary and sufficient condition such that each Lie derivation on a triangular algebra has the standard form ( $\spadesuit$ ). We next extend it to the generalized matrix algebras context.

**Theorem 3.3.** *Let  $\Theta_{\text{Lied}}$  be a Lie derivation of a generalized matrix algebra  $\mathcal{G}$ . Then  $\Theta_{\text{Lied}}$  is of the standard form ( $\spadesuit$ ) if and only if there exist linear mappings  $l_A : A \rightarrow \mathcal{Z}(A)$  and  $l_B : B \rightarrow \mathcal{Z}(B)$  satisfying*

- (1)  $p_A = \delta_1 - l_A$  is a derivation on  $A$ ,  $l_A([a, a']) = 0$ ,  $l_A(mn) = \delta_4(nm)$ ,  $l_A(a)m = m\mu_1(a)$  and  $nl_A(a) = \mu_1(a)n$ .
- (2)  $p_B = \mu_4 - l_B$  is a derivation on  $B$ ,  $l_B([b, b']) = 0$ ,  $l_B(nm) = \mu_1(mn)$ ,  $l_B(b)n = n\delta_4(b)$  and  $ml_B(b) = \delta_4(b)m$ .

*Proof.* Let us first prove the necessity. Suppose that  $\Theta_{\text{Lied}} = \delta + h$ , where  $\delta$  is a derivation and  $h$  maps  $\mathcal{G}$  into  $\mathcal{Z}(\mathcal{G})$ . Then by Lemma 3.2, there exist linear mappings  $l_A : A \rightarrow A$  and  $l_B : B \rightarrow B$  such that  $p_A = \delta_1 - l_A$  is a derivation of  $A$  and  $p_B = \mu_4 - l_B$  is a derivation of  $B$ . This gives

$$h \left( \begin{bmatrix} a & m \\ n & b \end{bmatrix} \right) = \begin{bmatrix} l_A(a) + \delta_4(b) & 0 \\ 0 & \mu_1(a) + l_B(b) \end{bmatrix} \in \mathcal{Z}(\mathcal{G}), \forall \begin{bmatrix} a & m \\ n & b \end{bmatrix} \in \mathcal{G}.$$

By Lemma 3.1 we know that  $\delta_4(b) \in \mathcal{Z}(A)$ ,  $\mu_1(a) \in \mathcal{Z}(B)$  for all  $b \in B$  and  $a \in A$ . Then the structure of  $\mathcal{Z}(\mathcal{G})$  implies that  $l_A$  maps into  $\mathcal{Z}(A)$  and  $l_B$  maps into  $\mathcal{Z}(B)$ . Furthermore,  $l_A(a)m = m\mu_1(a)$ ,  $nl_A(a) = \mu_1(a)n$ ,  $l_B(b)n = m\delta_4(b)$  and  $ml_B(b) = \delta_4(b)m$  are also follow.

Note that  $h$  is also a Lie derivation of  $\mathcal{G}$ . In view of Lemma 3.1 we have that  $l_A(mn) = \delta_4(nm)$  and  $l_B(nm) = \mu_1(mn)$  for all  $m \in M, n \in N$ . Let us choose  $G_1 = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$  and  $G_2 = \begin{bmatrix} a' & 0 \\ 0 & 0 \end{bmatrix}$  and take them into

$$h([G_1, G_2]) = [h(G_1), G_2] + [G_1, h(G_2)]. \quad (3.1)$$

It is not difficult to calculate that

$$h([G_1, G_2]) = \begin{bmatrix} l_A([a, a']) & 0 \\ 0 & \mu_1([a, a']) \end{bmatrix} \quad (3.2)$$

and

$$[h(G_1), G_2] + [G_1, h(G_2)] = \begin{bmatrix} [l_A(a), a'] + [a, l_A(a')] & 0 \\ 0 & 0 \end{bmatrix}. \quad (3.3)$$

It follows from  $l_A(a), l_A(a') \in \mathcal{Z}(A)$  that  $[l_A(a), a'] + [a, l_A(a')] = 0$ . Combining (3.2) with (3.3) yields that  $l_A([a, a']) = 0$ . Similarly, if we take  $G_1 = \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}$  and  $G_2 = \begin{bmatrix} 0 & 0 \\ 0 & b' \end{bmatrix}$  in (3.1), then  $l_B([b, b']) = 0$  will be obtained.

Let us see the sufficiency. Let us take

$$\delta \left( \begin{bmatrix} a & m \\ n & b \end{bmatrix} \right) = \begin{bmatrix} p_A(a) - mn_0 - m_0n & am_0 - m_0b + \tau_2(m) \\ n_0a - bn_0 + \nu_3(n) & n_0m + nm_0 + p_B(b) \end{bmatrix}, \forall \begin{bmatrix} a & m \\ n & b \end{bmatrix} \in \mathcal{G}$$

and

$$h \left( \begin{bmatrix} a & m \\ n & b \end{bmatrix} \right) = \begin{bmatrix} l_A(a) + \delta_4(b) & 0 \\ 0 & \mu_1(a) + l_B(b) \end{bmatrix}, \forall \begin{bmatrix} a & m \\ n & b \end{bmatrix} \in \mathcal{G}.$$

It is easy to verify that  $\delta$  is a derivation of  $\mathcal{G}$  and  $h$  maps into  $\mathcal{Z}(\mathcal{G})$ . For arbitrary  $G = \begin{bmatrix} a & m \\ n & b \end{bmatrix}$  and  $G' = \begin{bmatrix} a' & m' \\ n' & b' \end{bmatrix}$ , we have

$$h([G, G']) = \begin{bmatrix} l_A(x+u) + \delta_4(v+y) & 0 \\ 0 & \mu_1(x+u) + l_B(v+y) \end{bmatrix},$$

where  $x = [a, a']$ ,  $y = [b, b']$ ,  $u = mn' - m'n$  and  $v = nm' - n'm$ . Note that condition (1) implies that  $l_A(x+u) + \delta_4(v+y) = 0$  and condition (2) implies that  $\mu_1(x+u) + l_B(v+y) = 0$ . Therefore  $h([G, G']) = 0$ .  $\square$

The following corollary provides us a sufficient condition which enable each Lie derivation of  $\mathcal{G}(A, M, N, B)$  to be standard, where  $M$  is faithful as left  $A$ -module and as right  $B$ -module.

**Corollary 3.4.** *Let  $\mathcal{G} = \begin{bmatrix} A & M \\ N & B \end{bmatrix}$  be a generalized matrix algebra. Suppose that  $M$  is faithful as a left  $A$ -module and is also faithful as a right  $B$ -module. If  $\mathcal{Z}(A) = \pi_A(\mathcal{Z}(\mathcal{G}))$  and  $\mathcal{Z}(B) = \pi_B(\mathcal{Z}(\mathcal{G}))$ , then every Lie derivation of  $\mathcal{G}$  has the standard form  $(\spadesuit)$ .*

*Proof.* Let  $\Theta$  be a Lie derivation of  $\mathcal{G}$  with the form described in Lemma 3.1. Then it follows from  $\mathcal{Z}(A) = \pi_A(\mathcal{Z}(\mathcal{G}))$  and  $\mathcal{Z}(B) = \pi_B(\mathcal{Z}(\mathcal{G}))$  and Lemma 3.1 that

$$h \left( \begin{bmatrix} a & m \\ n & b \end{bmatrix} \right) = \begin{bmatrix} \varphi^{-1}(\mu_1(a)) + \delta_4(b) & 0 \\ 0 & \mu_1(a) + \varphi(\delta_4(b)) \end{bmatrix} \in \mathcal{Z}(\mathcal{G}).$$

On the other hand, it is a direct computation that  $\Theta - h$  is a derivation of  $\mathcal{G}$ . This completes the proof.  $\square$

**Remark 3.5.** Du and Wang obtained one much more general version of Corollary 3.4 in [5]. We omit the details here.

**Corollary 3.6.** *Let  $\mathcal{U} = \begin{bmatrix} A & M \\ O & B \end{bmatrix}$  be a triangular algebra. Suppose that  $M$  is faithful as a left  $A$ -module and is also faithful as a right  $B$ -module. If  $\mathcal{Z}(A) = \pi_A(\mathcal{Z}(\mathcal{U}))$  and  $\mathcal{Z}(B) = \pi_B(\mathcal{Z}(\mathcal{U}))$ , then every Lie derivation of  $\mathcal{U}$  is of the standard form  $(\spadesuit)$ .*



Let us continue to develop Theorem 11 of [4] to the case of generalized matrix algebras. As in [4], we need to give two preliminary lemmas.

**Lemma 3.7.** *Let  $\delta_1$  be a Lie derivation of  $A$  and let  $\mu_1 : A \rightarrow \mathcal{Z}(B)$  and  $\tau_2 : M \rightarrow M$ ,  $\nu_3 : N \rightarrow N$  be linear mappings satisfying*

$$\tau_2(am) = a\tau_2(m) + \delta_1(a)m - m\mu_1(a)$$

and

$$\nu_3(na) = \nu_3(n)a + n\delta_1(a) - \mu_1(a)n.$$

Define  $G : A \times A \rightarrow A$  by

$$G(x, y) = \delta_1(xy) - x\delta_1(y) - \delta_1(x)y.$$

Then the following statements hold:

- (1)  $G(x, y) = G(y, x)$ ;
- (2)  $G(x, y)m = m\mu_1(xy) - xm\mu_1(y) - ym\mu_1(x)$ ;  
 $nG(x, y) = \mu_1(xy)n - \mu_1(y)nx - \mu_1(x)ny$ .
- (3) Let  $f(t) = \sum_{j=0}^k r_j t^j \in K[t]$  and  $x \in A$ . Then there exists  $a_x \in A$  such that

$$a_x m = m\mu_1(f(x)) - f'(x)m\mu_1(x)$$

for all  $m \in M$  and

$$na_x = \mu_1(f(x))n - \mu_1(x)nf'(x)$$

for all  $n \in N$ , where  $f'(t) = \sum_{j=1}^k jr_j t^{j-1}$ .

Moreover, if  $\delta_1$  is of the standard form, that is,  $\delta_1 = p_A + l_A$ , where  $p_A$  is a derivation of  $A$  and  $l_A$  maps into  $\mathcal{Z}(A)$ , then  $a_x = l_A(f(x)) - f'(x)l(x)$ .

*Proof.* The relation (1) and the first one of (2) can be obtained via (i) and (ii) of Lemma 9 in [4]. It suffices to prove the second equality of (2). In fact, we have from the relation  $\nu_3(na) = \nu_3(n)a + n\delta_1(a) - \mu_1(a)n$  that

$$\nu_3(n(xy)) = \nu_3(n)xy + n\delta_1(xy) - \mu_1(xy)n$$

and

$$\begin{aligned} \nu_3((nx)y) &= \nu_3(nx)y + nx\delta_1(y) - \mu_1(y)nx \\ &= \nu_3(n)xy + n\delta_1(x)y - \mu_1(x)ny \end{aligned}$$

Then comparing the above two equalities gives the required result.

In order to prove (3), it is enough to consider  $f(t) = t^k$ , where  $k = 0, 1, 2, \dots$ . For  $k = 0$ , we can take  $a_x = \delta_1(1)$ , which is due to the conditions (4) and (5) of Lemma 3.1. For  $k > 0$ , let us take

$$a_x = \sum_{j=1}^{k-1} x^{k-1-j} G(x^j, x).$$

Then the first equality of (2) implies that

$$a_x m = m\mu_1(f(x)) - f'(x)m\mu_1(x).$$

The second equality of (2) implies that

$$na_x = \mu_1(f(x))n - \mu_1(x)nf'(x).$$

□

**Lemma 3.8.** Assume that  $\delta_1 = p_A + l_A$ , where  $p_A$  is a derivation and  $l_A(a) \in \mathcal{Z}(A)$ ,  $l_A([a, a']) = 0$  for all  $a, a' \in A$ . Let

$$V_A = \{a \in A \mid l_A(a)m = m\mu_1(a), \quad nl_A(a) = \mu_1(a)n \quad \forall m \in M, \forall n \in N\}.$$

Then  $V_A$  is a subalgebra of  $A$  satisfying the following conditions.

- (1)  $[x, y] \in V_A$  for all  $x, y \in A$ .
- (2) Let  $f(t) \in K[t]$  and  $x \in A$ . If  $f'(x) = 0$ , then  $f(x) \in V_A$ .
- (3)  $V_A$  contains all the idempotents of  $A$ .

*Proof.* For arbitrary  $x, y \in V_A$ , we have from (2) of Lemma 3.7 that

$$\begin{aligned} m\mu_1(xy) &= G(x, y)m + xm\mu_1(y) + ym\mu_1(x) \\ &= \delta_1(xy)m - x\delta_1(y)m - \delta_1(x)ym + xl_A(y)m + yl_A(x)m \\ &= p_A(xy)m + l_A(xy)m - xp_A(y)m \\ &\quad - xl_A(y)m - p_A(x)ym - l_A(x)ym + xl_A(y)m + yl_A(x)m \end{aligned}$$

Note that  $p_A$  is a derivation and  $l_A(x) \in \mathcal{Z}(A)$ . Then we obtain

$$m\mu_1(xy) = l_A(xy)m. \quad (3.4)$$

On the other hand,

$$\begin{aligned} \mu_1(xy)n &= nG(x, y) + \mu_1(y)nx + \mu_1(x)ny \\ &= n\delta_1(xy) - nx\delta_1(y) - n\delta_1(x)y + nl_A(y)x + nl_A(x)y \\ &= np_A(xy) + nl_A(xy) - nxp_A(y) \\ &\quad - nxl_A(y) - np_A(x)y - nl_A(x)y + nl_A(y)x + nl_A(x)y \\ &= nl_A(xy), \end{aligned}$$

That is,

$$\mu_1(xy)n = nl_A(xy). \quad (3.5)$$

Combining (3.4) with (3.5) yields that  $V_A$  is a subalgebra of  $A$ .

We now prove the other three conditions. Clearly, (1) follows from that  $\mu_1$  annihilates all commutators.

In order to prove (2), let us take  $x \in A$  with  $f'(x) = 0$  and  $f(t) \in \mathcal{R}[t]$ . In view of (3) of Lemma 3.7, we know that there exists  $a_x \in A$  such that  $a_x m = m\mu_1(f(x))$  and  $na_x = \mu_1(f(x))n$ . Since  $\delta_1$  is of the standard form,  $a_x m = l_A(f(x))m$  and  $na_x = nl_A(f(x))$  by Lemma 3.7. Therefore  $l_A(f(x))m = m\mu_1(f(x))$  and  $nl_A(f(x)) = \mu_1(f(x))n$ . That is,  $f(x) \in V_A$ .

The proof of (3) is the same as that of (4) of Lemma 10 in [4]. For the convenience of readers, we copy it here. In fact, for arbitrary idempotent  $e \in A$ , let us put  $f(t) = 3t^2 - 2t^3$ . Then clearly,  $f'(e) = 0$ . Hence  $e = f(e) \in V_A$ .  $\square$

As in [4], we define  $W(X)$  to be the smallest subalgebra of an algebra  $X$  satisfying conditions (1)-(3) of Lemma 3.8. Then the following result is a direct consequence of Theorem 3.3.

**Corollary 3.9.** Let  $\mathcal{G} = \begin{bmatrix} A & M \\ N & B \end{bmatrix}$  be a generalized matrix algebra with zero bilinear pairings. If

- (1)  $W(A) = A$  and every Lie derivation of  $A$  is of the standard form ( $\spadesuit$ );
- (2)  $W(B) = B$  and every Lie derivation of  $B$  is of the standard form ( $\spadesuit$ ),

then each Lie derivation of  $\mathcal{G}$  is of the standard form ( $\spadesuit$ ).

*Proof.* Since the bilinear pairings are both zero, the assumption (1) implies the condition (1) of Theorem 3.3 and the assumption (2) implies the condition (2) of Theorem 3.3.  $\square$

**Corollary 3.10.** [4, Theorem 11] *Every Lie derivation of a triangular algebra  $\text{Tri}(A, M, B)$  is of the standard form if the following conditions hold:*

- (1)  $W(A) = A$  and every Lie derivation of  $A$  is of the standard form;
- (2)  $W(B) = B$  and every Lie derivation of  $B$  is of the standard form,

At the end of this section, let us illustrate an application of Corollary 3.9. We will construct a class of algebras with bilinear pairings being both zero, which is called *generalized one-point extension algebras* in our situation. Note that they are not triangular algebras. We observe that each Lie derivation of a generalized one-point extension algebra is of the standard form ( $\spadesuit$ ). Moreover, the standard decomposition is unique.

**Definition 3.11.** Let  $(\Gamma_0, \Gamma_1)$  be a finite quiver without oriented cycles and  $|\Gamma_0| \geq 2$ . Let  $\Gamma^*$  be a quiver whose vertex set is  $\Gamma_0$  and

$$\Gamma_1^* = \{\alpha^* : i \rightarrow j \mid \alpha : j \rightarrow i \text{ is an arrow in } \Gamma_1\}.$$

For a path  $p = \alpha_n \cdots \alpha_1$  in  $\Gamma$ , write the path  $\alpha_1^* \cdots \alpha_n^*$  in  $\Gamma^*$  by  $p^*$ . Given a set  $\rho$  of relations, denote by  $\Lambda = K(\Gamma, \rho)$ . Define the generalized one-point extension algebra  $E(\Lambda)$  to be the path algebra of the quiver  $(\Gamma_0, \Gamma_1 \cup \Gamma_1^*)$  with relations

$$\rho \cup \rho^* \cup \{\alpha\beta^* \mid \alpha, \beta \in \Gamma_1\} \cup \{\alpha^*\beta \mid \alpha, \beta \in \Gamma_1\}.$$

In order to study Lie derivations of  $E(\Lambda)$ , we need the following lemmas.

**Lemma 3.12.**  $W(E(\Lambda)) = E(\Lambda)$ .

*Proof.* According to the definition,  $W(E(\Lambda))$  contains all idempotents  $e_i$ . Furthermore, for arbitrary arrow  $\alpha$  with  $e(\alpha) = j$ , the fact  $\alpha = [e_j, \alpha]$  implies that  $\alpha \in W(E(\Lambda))$ . With the same reason,  $\alpha^* \in W(E(\Lambda))$  and then  $W(E(\Lambda)) = E(\Lambda)$ .  $\square$

Since  $\Gamma$  is a quiver without oriented cycles, we can take a source  $i$  in  $\Gamma$ . Let  $e_i$  be the corresponding idempotent in  $E(\Lambda)$ . Then  $E(\Lambda)$  is isomorphic to a generalized matrix algebra  $\mathcal{G} = \begin{bmatrix} A & M \\ N & B \end{bmatrix}$  with  $A \simeq E(\Lambda')$ , where the quiver  $(\Gamma', \rho')$  of  $\Lambda'$  is obtained via removing the vertex  $i$  and the relations starting at  $i$ . Moreover, in view of the construction of  $E(\Lambda)$  we know that the bilinear pairings are both zero.

**Lemma 3.13.** *An additive mapping  $\Theta_d$  is a derivation of  $E(\Lambda)$  if and only if  $\Theta_d$  has the form*

$$\Theta_d \left( \begin{bmatrix} a & m \\ n & b \end{bmatrix} \right) = \begin{bmatrix} \delta_1(a) & am_0 - m_0b + \tau_2(m) \\ n_0a - bn_0 + \nu_3(n) & 0 \end{bmatrix},$$

$$\forall \begin{bmatrix} a & m \\ n & b \end{bmatrix} \in \mathcal{G},$$

where  $m_0 \in M, n_0 \in N$  and

$$\delta_1 : A \longrightarrow A, \quad \tau_2 : M \longrightarrow M, \quad \nu_3 : N \longrightarrow N$$

are all  $\mathcal{R}$ -linear mappings satisfying the following conditions:

- (1)  $\delta_1$  is a derivation of  $A$ ;
- (2)  $\tau_2(am) = a\tau_2(m) + \delta_1(a)m$  and  $\tau_2(mb) = \tau_2(m)b$ ;

$$(3) \quad \nu_3(na) = \nu_3(n)a + n\delta_1(a) \text{ and } \nu_3(bn) = b\nu_3(n).$$

*Proof.* Since the bilinear pairings are both zero, by Lemma 3.2, we only need to show that  $\mu_4 = 0$ . But this is clear by condition (2) of Lemma 3.2.  $\square$

**Lemma 3.14.** *Every derivation  $\Theta$  of  $E(\Lambda)$  with  $\text{Im}(\Theta) \in \mathcal{Z}(E(\Lambda))$  is zero.*

*Proof.* If there exists a nonzero derivation  $\Theta$  of  $E(\Lambda)$  with  $\text{Im}(\Theta) \in \mathcal{Z}(E(\Lambda))$ , then it follows from Lemma 3.13 that  $\delta_1 \neq 0$  and  $\text{Im}(\delta_1) \in \mathcal{Z}(A)$ . Repeating this process continuously, we eventually get that there exists a nonzero derivation  $f$  of  $K$ . However, this is impossible.  $\square$

**Proposition 3.15.** *Each Lie derivation of  $E(\Lambda)$  is of the standard form  $(\spadesuit)$ . Moreover, the standard decomposition is unique.*

*Proof.* Obviously,  $W(K) = K$ . If  $|\Gamma'_0| = 1$ , then  $A \simeq K$  and hence  $W(A) = A$ . If  $|\Gamma'_0| > 1$ , then  $A$  can be viewed as a generalized one-point extension too. Thus  $W(A) = A$ . By Corollary 3.9, if each Lie derivation of  $A$  has the standard form  $(\spadesuit)$ , then so is  $E(\Lambda)$ . Note that  $\Gamma$  is a finite quiver. Repeat the above process with finite times, we arrive at the algebra  $K$ . Of course, each Lie derivation of  $K$  is of the standard form. This implies that each Lie derivation of  $E(\Lambda)$  is standard. The uniqueness of the standard decomposition is due to Corollary 3.14.  $\square$

#### 4. LIE DERIVATIONS OF DUAL EXTENSIONS

**Lemma 4.1.** *Let  $\Gamma$  be a finite quiver without oriented cycles,  $A = K(\Gamma, \rho)$  and  $\mathcal{D}(A)$  be the dual extension of  $A$ . Then  $\mathcal{D}(A) = W(\mathcal{D}(A))$ .*

*Proof.* If  $\Gamma$  only contains one vertex, then the algebra  $\mathcal{D}(A)$  is trivial. That is,  $\mathcal{D}(A) \simeq K$ . In this case  $W(\mathcal{D}(A)) = \mathcal{D}(A)$ .

Now suppose that the number of vertices in  $\Gamma$  is not less than 2. It follows from the condition (3) in the definition of  $W(\mathcal{D}(A))$  that all the trivial paths are contained in  $W(\mathcal{D}(A))$ . On the other hand,  $\Gamma$  is a quiver without oriented cycles. Then for an arbitrary arrow  $\alpha \in \Gamma$ , we have  $\alpha = [\alpha, s(\alpha)]$ , which is due to the fact  $\alpha s(\alpha) = \alpha$  and  $s(\alpha)\alpha = 0$ . The condition (1) of the definition of  $W(\mathcal{D}(A))$  shows that  $\alpha \in W(\mathcal{D}(A))$ . Similarly, it can be proved that  $\alpha^* \in W(\mathcal{D}(A))$ . Therefore all paths are contained in  $W(\mathcal{D}(A))$ . Therefore  $\mathcal{D}(A) = W(\mathcal{D}(A))$ .  $\square$

**Lemma 4.2.** *Let  $\mathcal{D}(\Lambda)$  be the dual extension of a path algebra  $\Lambda = K(\Gamma, \rho)$ . Let  $i$  be a source in  $\Gamma$  and*

$$\mathcal{P}_i = \{p \in \mathcal{D} \mid s(p) = e(p) = i, p^2 = 0\}.$$

*Denote the vector space spanned by paths of  $\mathcal{P}_i$  by  $V$ . Assume that  $\Theta_{\text{Lied}}$  is a Lie derivation on  $\mathcal{D}(\Lambda)$ . Then  $\Theta_{\text{Lied}}(v)$  is in the center of  $\mathcal{D}(\Lambda)$  for all  $v \in V$ .*

*Proof.* It is easy to see that

$$\mathcal{D}(\Lambda) \simeq \begin{bmatrix} (1 - e_i)\mathcal{D}(\Lambda)(1 - e_i) & (1 - e_i)\mathcal{D}(\Lambda)e_i \\ e_i\mathcal{D}(\Lambda)(1 - e_i) & e_i\mathcal{D}(\Lambda)e_i \end{bmatrix}.$$

Then  $\Theta_{\text{Lied}}$  has the form described in Lemma 3.1. The condition (1) of Lemma 3.1 implies that  $\delta_4(p^*p) = 0$ . Thus  $\Theta_{\text{Lied}}(p^*p) = \mu_4(p^*p)$ . It follows from condition (2) of Lemma 3.1 that  $\mu_4(p^*p) = p^*q + q^*p$ . This shows that  $0m = m\mu_4(p^*p) = 0$  and  $n0 = \mu_4(p^*p)n = 0$  for all  $m \in M$  and  $n \in N$ . Note that  $B$  is commutative. Then  $\Theta_{\text{Lied}}(p^*p) \in \mathcal{Z}(\mathcal{D}(\Lambda))$ .  $\square$

Now we are in position to prove the main result of this paper.

**Theorem 4.3.** *Let  $\Gamma$  be a quiver without oriented cycles,  $\Lambda = K(\Gamma, \rho)$  and  $\mathcal{D}(\Lambda)$  be the dual extension of  $\Lambda$ . Then each Lie derivation on  $\mathcal{D}(\Lambda)$  is of the standard form ( $\spadesuit$ ).*

*Proof.* Let  $\Theta_{\text{Lied}}$  be a Lie derivation of  $\mathcal{D}(\Lambda)$ . Suppose that  $\mathcal{D}(\Lambda)$  as vector space has the decomposition  $\mathcal{D}(\Lambda) = V \oplus W$ . Define a linear mapping  $\Theta'$  by  $\Theta'(V) = 0$ ,  $\Theta'(W) = \Theta_{\text{Lied}}(W)$  and define a linear mapping  $\Delta'$  by  $\Delta'(V) = \Theta_{\text{Lied}}(V)$  and  $\Delta'(W) = 0$ . Clearly,  $\Theta_{\text{Lied}} = \Theta' + \Delta'$ . Furthermore, it follows from Lemma 4.2 that  $\text{Im}(\Delta') \subset \mathcal{Z}(\mathcal{D}(\Lambda))$ . This shows that  $\Theta'$  is also a Lie derivations on  $\mathcal{D}(\Lambda)$ . Clearly, if each Lie derivations of type  $\Theta'$  is of the standard form ( $\spadesuit$ ), then every Lie derivations of  $\mathcal{D}(\Lambda)$  has the standard form ( $\spadesuit$ ).

Assume that  $i$  is a source in  $\Gamma$  and  $e_i$  the corresponding idempotent in  $\mathcal{D}(\Lambda)$ . Let  $\Theta_{\text{Lied}}$  be a Lie derivation of  $\mathcal{D}(\Lambda)$  satisfying the condition  $\Theta_{\text{Lied}}(V) = 0$ . Then

$$\mathcal{D}(\Lambda) \simeq \begin{bmatrix} (1 - e_i)\mathcal{D}(\Lambda)(1 - e_i) & (1 - e_i)\mathcal{D}(\Lambda)e_i \\ e_i\mathcal{D}(\Lambda)(1 - e_i) & e_i\mathcal{D}(\Lambda)e_i \end{bmatrix}.$$

Let us first prove that for  $\mathcal{D}(\Lambda)$ , the condition (2) of Theorem 3.3 is satisfied. We have from the construction of  $\mathcal{D}(\Lambda)$  that  $e_i\mathcal{D}(\Lambda)e_i$  is an algebra with a basis  $\{p^*p \mid s(p) = i\}$ . Furthermore, if  $p, q$  are nontrivial, then  $(p^*p)(q^*q) = 0$ . Thus the algebra  $e_i\mathcal{D}(\Lambda)e_i$  is commutative. Let  $l_B$  be equal to  $\mu_4$ . Then  $l_B([b, b']) = 0$  for all  $b, b' \in e_i\mathcal{D}(\Lambda)e_i$ . Note that  $(1 - e_i)\mathcal{D}(\Lambda)e_i\mathcal{D}(\Lambda)(1 - e_i) = 0$ . That is,  $\Phi_{MN} = 0$ . We conclude that  $\mu_1(mn) = 0$  for all  $m \in M$  and  $n \in N$ . On the other hand, since  $\Theta(p^*p) = 0$  for all nontrivial path  $p$ , we arrive at  $\mu_4(p^*p) = 0$  and hence  $\mu_4(nm) = 0$ . Therefore  $l_B(nm) = \mu_1(mn)$ .

Let  $b = ke_i + v$ , where  $v \in V$ . It follows from the structure of  $M$  and  $\mathcal{D}(\Lambda)$  that

$$\tau_2(mb) = k\tau_2(me_i) = k\tau_2(m) = \tau_2(m)ke_i = \tau_2(m)(ke_i + v) = \tau_2(m)b.$$

Similarly, we can obtain that  $\nu_3(bn) = b\nu_3(n)$ . Then it follows from conditions (4) and (5) of Lemma 3.1 that  $l_B(b)n = n\delta_4(b)$  and  $ml_B(b) = \delta_4(b)m$ .

By the definition of dual extension, it is easy to check that

$$(1 - e_i)\mathcal{D}(\Lambda)(1 - e_i) \simeq \mathcal{D}(\Lambda'),$$

where  $\Lambda' = K(\Gamma', \rho')$ ,  $(\Gamma', \rho')$  is a quiver obtained by removing the vertex  $i$  and the relations starting at  $i$ . Clearly,  $\Gamma'$  has no oriented cycles. Then Lemma 4.1 implies that  $\mathcal{D}(\Lambda') = W(\mathcal{D}(\Lambda'))$ . Thus  $\Theta_{\text{Lied}}$  is of the standard form ( $\spadesuit$ ) if each Lie derivation on  $\mathcal{D}(\Lambda')$  is standard.

Note that  $\Gamma$  is a finite quiver. Repeating this process with finite times, we arrive at the algebra  $K$ . That is, if each Lie derivation of  $K$  is standard, then so is  $\mathcal{D}(\Lambda)$ . This completes the proof.  $\square$

**Corollary 4.4.** *Let  $\Theta_{\text{Lied}}$  be a Lie derivation of  $\mathcal{D}(\Lambda)$ . Then there exists a derivation  $D$  of  $\mathcal{D}(\Lambda)$  such that  $\Theta_{\text{Lied}}(x) = D(x)$  for all  $x = \sum_i k_i p_i \in \Lambda$ , where  $p_i$  are non-trivial paths.*

*Proof.* Let  $\Theta_{\text{Lied}}$  be a Lie derivation of  $\mathcal{D}(\Lambda)$ . We have from Theorem 4.3 that  $\Theta_{\text{Lied}}$  is of the standard form ( $\spadesuit$ ). Then  $\Theta_{\text{Lied}} = D + \Delta$ , where  $D$  is a derivation of  $\mathcal{D}(\Lambda)$  and  $\Delta(x) \in \mathcal{Z}(\mathcal{D}(\Lambda))$  for all  $x \in \mathcal{D}(\Lambda)$ . Note that  $\Delta$  is also a Lie derivation of  $\mathcal{D}(\Lambda)$ . Then for a path  $p$  with  $s(p) \neq e(p)$ , the fact  $p = [p, s(p)]$  gives

$$\Delta(p) = [\Delta(p), s(p)] + [p, \Delta(s(p))].$$

It follows from the image of  $\Delta$  being in  $\mathcal{Z}(\mathcal{D}(\Lambda))$  that  $\Delta(p) = 0$ . Moreover, let  $p$  be a nontrivial path with  $s(p) = e(p)$ . By the construction of  $\mathcal{D}(\Lambda)$ ,  $p$  is of the form  $x^*x$ , where  $x$  is a nontrivial path in  $\Gamma$ . Therefore

$$\Delta(p) = \Delta(x^*x) = \Delta([x^*, x]) = [\Delta(x^*), x] + [x^*, \Delta(x)] = 0.$$

Then for all  $x = \sum_i k_i \bar{p}_i \in \Lambda$ , where  $p_i$  are non-trivial paths, we have  $\Theta_{\text{Lied}}(x) = D(x)$ .  $\square$

Let us address problem of whether the standard decomposition of arbitrary Lie derivation of  $\mathcal{D}(\Lambda)$  is unique. Fortunately, the answer is positive. In order to give the answer, we are forced to characterize the center of  $\mathcal{D}(\Lambda)$ .

**Lemma 4.5.** *Let  $\Gamma$  be a connected quiver with  $|\Gamma_0| \geq 2$ . Then the elements in  $\mathcal{Z}(\mathcal{D}(\Lambda))$  are of the form*

$$k + \sum_{e(p)=s(p), p^2=0} k_p p.$$

*Proof.* Assume that

$$x = \sum_{i \in \Gamma_0} k_i e_i + \sum_{s(p) \neq e(p)} k_p p + \sum_{s(p)=e(p), p^2=0} k_p p \in \mathcal{Z}(\mathcal{D}(\Lambda)).$$

Applying the fact  $e_t x = x e_t$  yields that

$$\sum_{t=s(p) \neq e(p)} k_p p = \sum_{t=e(p) \neq s(p)} k_p p.$$

This implies that for all paths  $p$  with  $s(p) \neq e(p)$ , we have  $k_p = 0$  if  $s(p) = t$  or  $e(p) = t$ . Since  $t$  is arbitrary, the coefficients of all paths  $p$  with  $s(p) \neq e(p)$  are zero.

Let  $\alpha$  be an arrow in  $\Gamma_1$  with  $e(\alpha) = j$  and  $s(\alpha) = t$ . In view of  $\alpha x = x \alpha$ , we know that  $k_j = k_t$ . Note that  $\Gamma$  is a connected quiver. Thus  $k_i = k$  for all  $i \in \Gamma_0$ , where  $k \in K$ . This completes the proof.  $\square$

**Lemma 4.6.** *Let  $D$  be a derivation of  $\mathcal{D}(\Lambda)$ . If  $\text{Im}(D) \subset \mathcal{Z}(\mathcal{D}(\Lambda))$ , then  $D = 0$ .*

*Proof.* Let  $D$  be a derivation of  $\mathcal{D}(\Lambda)$  with  $\text{Im}(D) \subset \mathcal{Z}(\mathcal{D}(\Lambda))$ . Clearly,  $D$  is also a Lie derivation of  $\mathcal{D}(\Lambda)$ . By the proof of Corollary 4.4, we have  $D(p) = 0$  for all nontrivial path  $p$ . We now prove  $D(e_i) = 0$  for all  $i \in \Gamma_0$ . According to Corollary 4.5, we can assume that  $D(e_i) = k_i + \sum_{e(p)=s(p), p^2=0} k_p^i p$ . Note that  $e_i$  is an idempotent. By the definition of derivation, it is easy to verify that  $k_i = 0$  and  $k_p^i = 0$  for paths  $p$  with  $s(p) = i$ . Furthermore, suppose there exists some  $p$  with nonzero coefficient in  $D(e_i)$ . Let  $s(p) = j \neq i$ . Then  $D(e_i e_j) = 0$ . On the other hand,  $D(e_i e_j) = D(e_i) e_j + e_i D(e_j) \neq 0$ , which is a contradiction. This implies that  $D = 0$ .  $\square$

As a direct consequence of Lemma 4.6 we immediately get

**Proposition 4.7.** *Let  $\Theta_{\text{Lied}}$  be a Lie derivation of a dual extension algebra  $\mathcal{D}(\Lambda)$ . Then the standard decomposition of  $\Theta_{\text{Lied}}$  is unique.*

**Remark 4.8.** On one hand, we know that a Lie derivation of dual extension algebra can be uniquely expressed as the sum of a derivation and a linear mapping annihilating all commutators with images in the center of the algebra. On the other hand, the sum of a derivation and a linear mapping annihilating all commutators

with images in the center is clearly a Lie derivation. In this sense, the forms of Lie derivation on dual extensions are thoroughly characterized at all.

Now let us give an example of a Lie derivation which is not a derivation.

**Example 4.9.** Let  $\Gamma$  be the following quiver

$$\begin{array}{ccccc} 1 & \xrightarrow{\alpha} & 2 & \xrightarrow{\beta} & 3 \end{array}$$

with relation  $\beta\alpha$  and  $\Lambda = K(\Gamma, \rho)$ . Let  $\mathcal{D}(\Lambda)$  be the dual extension of the path algebra  $\Lambda$ . Define a linear mapping  $\Theta_{\text{Lied}}$  on  $\mathcal{D}(\Lambda)$  by

$$\begin{aligned} \Theta_{\text{Lied}}(e_1) &= k_1 + \alpha^* \alpha; & \Theta_{\text{Lied}}(e_2) &= k_2 + \beta^* \beta; & \Theta_{\text{Lied}}(e_3) &= k_3; \\ \Theta_{\text{Lied}}(\alpha) &= \alpha; & \Theta_{\text{Lied}}(\alpha^*) &= \alpha^*; & \Theta_{\text{Lied}}(\beta) &= \beta^*; \\ \Theta_{\text{Lied}}(\beta^*) &= \beta^*; & \Theta_{\text{Lied}}(\alpha^* \alpha) &= 2\alpha^* \alpha; & \Theta_{\text{Lied}}(\beta^* \beta) &= 2\beta^* \beta \end{aligned}$$

Then a direct computation shows that  $\Theta_{\text{Lied}}$  is a Lie derivation of  $\mathcal{D}(\Lambda)$  but not a derivation.

Moreover, we give the standard decompositions of  $\Theta_{\text{Lied}}$  here. Define a linear mapping  $\Delta$  on  $\mathcal{D}(\Lambda)$  by

$$\Delta(e_1) = k_1 + \alpha^* \alpha, \quad \Delta(e_2) = k_2 + \beta^* \beta, \quad \Delta(e_3) = k_3$$

and let  $D = \Theta_{\text{Lied}} - \Delta$ , then  $\Theta_{\text{Lied}} = D + \Delta$  is the standard decomposition of  $\Theta$ .

**Acknowledgement** The first author would like to express his sincere thanks to Chern Institute of Mathematics of Nankai University for the hospitality during his visit. He also acknowledges Prof. Chengming Bai for his kind consideration and warm help.

## REFERENCES

- [1] J. Alaminos, M. Brešar and A. R. Villena, *The strong degree of von Neumann algebras and the structure of Lie and Jordan derivations*, Math. Proc. Camb. Phil. Soc., **137** (2004), 441-463.
- [2] M. Auslander, I. Reiten, S. Smalø, *Representation Theory of Artin Algebras*, Cambridge Studies in Advanced Mathematics, vol. **36**, Cambridge University Press, Cambridge (1995).
- [3] D. Benkovič, *Lie derivations on triangular matrices*, Linear Multilinear Algebra, **55** (2007), 619-626.
- [4] W. S. Cheung, *Lie derivations of triangular algebras*, Linear Multilinear Algebra, **51** (2003), 299-310.
- [5] Y. Du and Y. Wang, *Lie derivations of generalized matrix algebras*, Linear Algebra Appl., **437** (2012), 2719-2726.
- [6] B. M. Deng and C. C. Xi, *Quasi-hereditary algebras which are dual extensions of algebras*, Comm. Algebra, **22** (1994), 4717-4735.
- [7] B. M. Deng and C. C. Xi, *Quasi-hereditary algebras which are twisted double incidence algebras of posets*, Beiträge Algebra Geom., **36** (1995), 37-71.
- [8] B. M. Deng and C. C. Xi, *Ringel duals of quasi-hereditary algebras*, Comm. Algebra **24** (1996), 2825-2838.
- [9] L. Guo and F. Li, *On the Lie algebra of differential operators on a path algebra*, arXiv:1010.1980
- [10] R. S. Irving, *BGG-algebras and the BGG reciprocity principle*, J. Algebra **135** (1990), 363-380.
- [11] Y. -B. Li and F. Wei, *Semi-centralizing maps of generalized matrix algebras*, Linear Algebra Appl., **436** (2012), 1122-1153.
- [12] Y. -B. Li and F. Wei, *Jordan derivations and Lie derivations of path algebras*, arXiv: 1203.4925.
- [13] Y. -B. Li and F. Wei, *Jordan derivations of dual extensions*, In Preparation.

- [14] Y. -B. Li, L. van Wyk and Feng Wei, *Jordan derivations and antiderivations of generalized matrix algebras*, Operators and Matrices, In Press.
- [15] W. S. Martindale, Lie derivations of primitive rings, Michigan Math. J., **11** (1964), 183-187.
- [16] K. Morita, *Duality for modules and its applications to the theory of rings with minimum condition*, Sci. Rep. Tokyo Kyoiku Diagaku Sect., **A6** (1958), 83-142.
- [17] S. Koenig and C. C. Xi, *Strong symmetry defined by twisting modules, applied to quasi-hereditary algebras with triangular decomposition and vanishing radical cube* Comm. Math. Phys., **197** (1998), 427-441.
- [18] C. C. Xi, *Quasi-hereditary algebras with a duality*, J. Reine Angew. Math., **449**(1994), 201-215.
- [19] C. C. Xi, *Global dimensions of dual extension algebras*, Manuscripta Math., **88**(1995), 25-31.
- [20] C. C. Xi, *Twisted doubles of algebras I: Deformations and the Jones index*, CMS Conf. Proc., **24** (1998), 513-523.
- [21] Z. -K. Xiao and F. Wei, *Commuting mappings of generalized matrix algebras*, Linear Algebra Appl., **433** (2010), 2178-2197.
- [22] Z. -K. Xiao and F. Wei, *Lie tripple derivations of triangular algebras*, Linear Algebra Appl., **437** (2012), 1234-1249.

LI: SCHOOL OF MATHEMATICS AND STATISTICS, NORTHEASTERN UNIVERSITY AT QINHUANGDAO,  
 QINHUANGDAO, 066004, P.R. CHINA  
*E-mail address:* liyanbo707@163.com

WEI: SCHOOL OF MATHEMATICS, BEIJING INSTITUTE OF TECHNOLOGY, BEIJING, 100081, P. R.  
 CHINA  
*E-mail address:* daoshuo@hotmail.com